ISRAEL JOURNAL OF MATHEMATICS 75 (1991), 341–370

AMENABILITY, KAZHDAN'S PROPERTY AND PERCOLATION FOR TREES, GROUPS AND EQUIVALENCE RELATIONS*

BY

SCOT ADAMS

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

AND

RUSSELL LYONS

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

ABSTRACT

We prove amenability for a broad class of equivalence relations which have trees associated to the equivalence classes. The proof makes crucial use of percolation on trees. We also discuss related concepts and results, including amenability of automorphism groups. A second main result is that no discrete subgroup of the automorphism group of a tree is isomorphic to the fundamental group of any closed manifold M admitting a nontrivial connection-preserving, volume-preserving action of a noncompact, simply connected, almost simple Lie group having Kazhdan's property (T). The technique of proof also shows that M does not admit a hyperbolic structure.

1. Introduction

Amenability is a notion originating in analysis on groups which has come to find broad applicability and relevance [P]. One way to think of amenability is as a tool of classification. Another is as a notion of smallness; e.g., if a group is finite, compact, or abelian, then it is amenable. In Section 2, we review the (sometimes multiple) definitions of amenability for groups, group actions, measured equivalence relations, Borel equivalence relations, and classes of graphs. Whereas for

*The research of both authors was supported by NSF Postdoctoral Fellowships. The second author was also supported by an Alfred P. Sloan Foundation Research Fellowship.
Received June 25, 1991

groups, amenability can be defined equivalently in terms of invariant means or in terms of fixed points, for group actions these definitions split into distinct and complementary notions. Both have been called "amenable" before; we propose to call one "co-amenable." Proposition 2.4 shows one way in which these notions are complementary. The notion of amenability for classes of graphs (or other structures) is a very recent one due to Kechris [Kec]. We find it particularly appealing since it formalizes whether "canonical" choices and constructions are possible.

If an equivalence relation has a graph associated to each equivalence class, then amenability of the equivalence relation is related to the geometry of the graphs [A]. In particular, if these graphs are trees which are small in some sense, then we obtain amenability [A], [DK]. In Theorem 4.4 and Corollary 4.5, we extend all previous results in this direction by showing that the class of trees of branching number 1 [L1] – i.e., trees whose boundary has Hausdorff dimension 0 – is amenable. This is accomplished by means of percolation processes on such trees, which is the first connection of which we are aware between percolation and amenability. It is surprising that although the class of trees of branching number 1 can be defined via random walks and although Kesten [Kes1, Kes2] and others since have shown the close tie between random walks and amenability, we were able to find no proof of our result which used random walks.

Before demonstrating this result, we present a simple proof of a theorem of Nebbia [N] characterizing amenable automorphism groups of trees and of a partial extension due to Woess [Woe] for graphs.

A property in some ways complementary to amenability for groups is that of Kazhdan's property (T). Our second main result concerns manifolds with such fundamental groups and their possible actions on trees. Namely, let G be a connected, noncompact, almost simple Lie group with finite fundamental group. Suppose that G has property (T). (All definitions will be recalled in Section 5.) Let M be a closed, real analytic manifold with a real analytic connection and a smooth volume. Assume that G acts real analytically on M preserving the connection and volume and that the action is nontrivial. Let Γ denote the fundamental group of M. Then Γ cannot act properly on any tree. By [S, Section 4, Theorem 7], this extends a theorem of Zimmer [Z4, Theorem 7.1] that Γ is not isomorphic to the amalgam of two finite groups over a common subgroup.

We are grateful to Alexander Kechris for discussions of amenability and metamathematics.

2. Generalities on Amenability

If G is a locally compact group, we denote by ρ_G a right Haar measure on G and $L^\infty(G) := L^\infty(G, \rho_G; \mathbf{R})$. Here and throughout this paper, all Banach spaces will be real. If A is a linear subspace of $L^\infty(G)$ containing the constant function $\mathbf{1}$, we say that a linear functional m on A is a **mean** if $m(\mathbf{1}) = 1$ and $m \geq 0$ (i.e., $0 \leq f \in A \Rightarrow m(f) \geq 0$). If $g \in G$ and $f : G \to \mathbf{R}$, we write $R_g f : x \mapsto f(xg)$. We say that a mean m on A is **invariant** if $m(R_g f) = m(f)$ for all $f \in A$ and $g \in G$.

Definition: A locally compact group G is **amenable** if there is an invariant mean on $L^\infty(G)$.

There are many properties equivalent to amenability, some of which we recall now. Denote the space of bounded continuous functions on G by $C_b(G)$.

PROPOSITION 2.1: *A locally compact group G is amenable iff there is an invariant mean on $C_b(G)$.*

See [P, Corollary 1.10] for a proof. The easiest way to see that amenability is related to the growth of the group, as alluded to in the introduction, is via Følner's Theorem [P, Theorem 4.10]:

THEOREM 2.2: *A locally compact group G is amenable iff for every $\varepsilon > 0$ and every compact $C \subseteq G$, there is a nonnull compact $K \subseteq G$ such that*

$$\forall g \in C \quad \frac{\rho_G(Kg \triangle K)}{\rho_G(K)} < \varepsilon.$$

We say that G has **subexponential growth** if every compact neighborhood K of the identity satisfies

$$\liminf_{n\to\infty} \frac{\rho_G(K^{n+1})}{\rho_G(K^n)} = 1.$$

An evident consequence of Følner's Theorem is that such groups are amenable.

Another equivalent property that we will use is Day's Fixed-Point Theorem on affine actions. If A is a compact convex subset of a locally convex topological vector space, a map $T : A \to A$ is called **affine** if

$$\forall \alpha \in [0,1]\ \forall x, y \in A \quad T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y).$$

We say that G **acts affinely** on A if there is a right action of G on A by affine maps.

THEOREM 2.3 [P, THEOREM 2.24]: *A locally compact group G is amenable iff every jointly continuous affine action of G on a compact affine space has a fixed point.*

We turn now to amenability and co-amenability for group actions (cf. [Gre]). Let (S,μ) be a measure space. A right action of G on S by μ-measurable bijections is called **nonsingular** if

$$\forall E \subseteq S \ \forall g \in G \quad \mu(E) = 0 \Rightarrow \mu(Eg) = 0.$$

For $g \in G$ and $f : S \to \mathbf{R}$, we write $R_g f : s \mapsto f(sg)$. Thus, a nonsingular action of G on (S,μ) induces a left action of G on $L^\infty(S,\mu)$ by $R_\bullet$; we write $L^\infty_G(S,\mu)$ for the subspace of $f \in L^\infty(S,\mu)$ for which the map $g \mapsto R_g f$ is continuous from G to $L^\infty(S,\mu)$ (the latter having the usual norm topology). A **mean** on $L^\infty_G(S,\mu)$ is a linear functional m satisfying $m(\mathbf{1}) = 1$ and $m \geq 0$; it is **invariant** if $m(R_g f) = m(f)$ for all $g \in G$ and $f \in L^\infty_G(S,\mu)$.

Definition: A nonsingular action of a locally compact group G on a measure space (S,μ) is **co-amenable** if there is an invariant mean on $L^\infty_G(S,\mu)$.

Note that if G is discrete, then $L^\infty_G(S,\mu) = L^\infty(S,\mu)$. Thus, if a G-action is co-amenable when G has the discrete topology, it is also co-amenable in any other topology. The left action of G on $L^\infty(S,\mu)$ induces a jointly continuous right action on $L^\infty_G(S,\mu)^*$. This action is affine on the compact convex set of means (in the weak* topology), whence, by Day's Theorem, G is amenable iff every nonsingular G-action is co-amenable.

As mentioned in the introduction, the property of group actions analogous to the fixed point property of amenable groups is not equivalent to the above definition. Let G be a second countable locally compact group with a nonsingular action on a standard Borel space S with Borel probability measure μ, i.e., on a standard measure space. The following definitions are taken from [Z2]. Suppose that M is a topological group. A Borel function $\alpha : S \times G \to M$ is called a **cocycle** if

$$\forall g,h \in G \quad \text{for } \mu\text{-a.e. } s \in S \quad \alpha(s,g)\alpha(sg,h) = \alpha(s,gh).$$

Let E be a separable Banach space and $\mathrm{Iso}(E)$ be the group of isometric isomorphisms of E with the strong operator topology. Let E_1^* be the closed unit ball of the dual of E with the weak* topology and $H(E_1^*)$ the group of homeomorphisms of E_1^* with the topology of uniform convergence. Every cocycle

$\alpha : S \times G \to \mathrm{Iso}(E)$ induces an **adjoint cocycle** $\alpha^* : S \times G \to H(E_1^*)$ by $\alpha^*(s,g) := (\alpha(s,g)^{-1})^*$. A map $s \mapsto A_s$ assigning to each $s \in S$ a non-empty compact convex set $A_s \subseteq E_1^*$ is called a **Borel field** if $\{(s,a);\ a \in A_s\}$ is a Borel subset of $S \times E_1^*$. (Standard techniques show that this is the same as requiring the map $s \mapsto A_s$ to be Borel from S to the space of compact subsets of E_1^* with the Hausdorff topology.) A Borel field $\langle A_\bullet \rangle$ is called **α-invariant** if

$$\forall g \quad \text{for } \mu\text{-a.e. } s \quad \alpha^*(s,g)A_{sg} = A_s;$$

in this case, we call $(\alpha, \langle A_\bullet \rangle)$ an **affine G-space over** (S,μ). A μ-measurable map $\varphi : S \to E_1^*$ is called a **section** of $\langle A_\bullet \rangle$ if for μ-a.e. s $\varphi(s) \in A_s$; the section is **α-invariant** if

$$\forall g \quad \text{for } \mu\text{-a.e. } s \quad \alpha^*(s,g)\varphi(sg) = \varphi(s).$$

Definition: A nonsingular action of a second countable locally compact group G on a standard measure space (S,μ) is **amenable** if every affine G-space over (S,μ) has an invariant section.

Zimmer ([Z2, Corollary 1.6] and [Z2, Theorem 2.1], where the assumption of ergodicity is never used) shows that G is amenable iff every nonsingular G-action (on a standard measure space) is amenable. The following proposition shows how co-amenability is complementary to amenability. It extends and clarifies [Z3, Proposition 4.3.3].

PROPOSITION 2.4: *Let G be a second countable locally compact group. Then G is amenable iff there is some nonsingular G-action on a standard measure space which is both amenable and co-amenable.*

Proof: Let G act nonsingularly on the standard measure space (S,μ). We have already seen that if G is amenable, then the action is both amenable and co-amenable. Now suppose that the action is both amenable and co-amenable. In order to show that G is amenable, it suffices, by [Z2, Proposition 1.5], to show that for every separable Banach space E, continuous homomorphism $\pi : G \to \mathrm{Iso}(E)$, and G-invariant compact convex set $A \subseteq E_1^*$, there is a G-fixed point in A. Consider the cocycle

$$\alpha(s,g)e := \pi(g)e \qquad (s \in S,\ g \in G,\ e \in E)$$

and the constant Borel field $s \mapsto A$ which is, of course, α-invariant. Let φ be an invariant section and let m be an invariant mean on $L^\infty_G(S,\mu)$. Set

$$f_e(s) = \langle e, \varphi(s)\rangle \qquad (s \in S,\ e \in E).$$

We claim that $f_e \in L^\infty_G(S,\mu)$ for all $e \in E$. Indeed, if $g \in G$, then because φ is invariant,

$$\begin{aligned}(R_g f_e)(s) = f_e(sg) &= \langle e, \varphi(sg)\rangle = \langle e, \alpha^*(s,g)^{-1}\varphi(s)\rangle \\ &= \langle e, \pi(g)^*\varphi(s)\rangle = \langle \pi(g)e, \varphi(s)\rangle = f_{\pi(g)e}(s),\end{aligned}$$

whence for $g, h \in G$,

$$|R_g f_e(s) - R_h f_e(s)| = |\langle \pi(g)e - \pi(h)e, \varphi(s)\rangle| \le \|\pi(g)e - \pi(h)e\|$$

in light of the fact that $\varphi(s) \in E_1^*$. By continuity of π, we see that $g \mapsto R_g f_e$ is continuous as claimed.

We may now define $a \in E^*$ by

$$\langle e, a\rangle := m(f_e) \qquad (e \in E).$$

The Hahn–Banach Theorem guarantees that $a \in A$. It remains to establish that $\forall g \in G \quad \pi(g)^* a = a$. Now for $e \in E$, we calculate

$$\langle e, \pi(g)^* a\rangle = \langle \pi(g)e, a\rangle = m(f_{\pi(g)e}) = m(R_g f_e) = m(f_e) = \langle e, a\rangle$$

using the observation above and the invariance of m. This completes the proof. □

We now proceed to the next level of generality, that of measured equivalence relations, also called equivalence spaces; we shall consider only countable equivalence relations. Thus, an **equivalence space** (S,μ,R) is a standard measure space (S,μ) with an equivalence relation $R \subseteq S \times S$ which is Borel and has the property that each equivalence class is countable. We denote the equivalence class of $s \in S$ by $[s]$. If $F : R \to \mathbf{R}$ and $s \in S$, we write $F_s : [s] \to \mathbf{R}$ for the function $F_s(t) := F(s,t)$. A **mean** on S/R is a map which assigns to each $[s]$ a mean $m_{[s]}$ on $\ell^\infty([s])$; it is said to be **μ-measurable** if whenever $F : R \to \mathbf{R}$ is bounded and Borel, the map $s \mapsto m_{[s]}(F_s)$ on S is μ-measurable.

Definition [CFW]*:* An equivalence space (S,μ,R) is **amenable** if there is a μ-measurable mean on S/R.

If G is a countable discrete group acting nonsingularly on a standard measure space (S,μ) by Borel automorphisms, then the orbits induce the equivalence space (S,μ,E_G), where $(s,t)\in E_G$ iff $s\in tG$. It is not true that co-amenability of the G-action implies amenability of E_G. However, amenability of the G-action (hence amenability of G) does ensure amenability of E_G. In order to show this, we require an equivalent definition of amenable equivalence space due to Zimmer [Z1]. This definition parallels that of amenable actions, requiring only an adjustment in the notion of cocycle.

Given an equivalence space (S,μ,R) and a topological group M, we call $\alpha : R\to M$ a **cocycle** if α is Borel and

$$\forall (s,t),(t,u)\in R \qquad \alpha(s,t)\alpha(t,u)=\alpha(s,u).$$

Again, if $M=$ Iso(E) for some separable Banach space E, the **adjoint cocycle** is defined by $\alpha^*(s,t):=(\alpha(s,t)^{-1})^*$. A Borel field $\langle A_\bullet\rangle$ is **α-invariant** if

$$\text{for } \mu\text{-a.e. } t \quad \forall s\in[t] \qquad \alpha^*(s,t)A_t=A_s;$$

in this case, we call $(\alpha,\langle A_\bullet\rangle)$ an **affine space over** (S,μ,R). A section φ is **α-invariant** if

$$\text{for } \mu\text{-a.e. } t \quad \forall s\in[t] \qquad \alpha^*(s,t)\varphi(t)=\varphi(s).$$

Definition: An equivalence space (S,μ,R) is **amenable (in the sense of Zimmer)** if every affine space over it has an invariant section.

That amenable group actions induce equivalence spaces which are amenable in Zimmer's sense is clear [Z1]. To show that an amenable equivalence space is amenable in Zimmer's sense is also fairly straightforward. The converse, however, is more difficult. See Appendix 1 for the proofs.

We shall, in fact, be proving amenability for equivalence relations in a stronger sense where no measure is present; such results have applications for every measure. These notions are due to Kechris [Kec], who used them to solve a problem of Slaman and Steel on definability. Given a (countable) Borel equivalence relation R on a standard Borel space S, a mean on S/R is called **universally measurable** if it is μ-measurable for every Borel probability measure, μ.

Definition: An equivalence relation (S,R) is **amenable** if there is a universally measurable mean on S/R.

Evidently, every equivalence space formed from an amenable equivalence relation is itself amenable. Unfortunately, the only ways known to produce nontrivial amenable equivalence relations depend on assuming the continuum hypothesis (CH). Nevertheless, if (S, R) is proved to be amenable using CH, then (S, μ, R) is still amenable without assuming CH, as shown by standard metamathematical arguments using forcing and Zimmer's definition of amenability; see Appendix 2.

Often, given an equivalence relation, there is a Borel assignment of structures, such as trees, to the equivalence classes. If these structures are drawn from a class with special properties, that alone may guarantee amenability. The only structures we shall consider are countable graphs, so we present the definitions from [A] and [Kec] in this context.

Definition: A **graphed equivalence relation** (S, R, Q) is an equivalence relation (S, R) with a symmetric Borel subrelation $Q \subseteq R$. A **graph** in (S, R, Q) is a graph whose vertex set is $[s]$ and edge set is $Q \restriction [s] \times [s]$ for some $s \in S$.

In order to define amenability and co-amenability for a class of graphs, we need to parametrize the collection of graphs. Let $\mathcal{G}$ be the Polish space ${}^{\omega}2 \times {}^{\omega\times\omega}2$. We regard elements $\gamma \in \mathcal{G}$ as pairs (V, Q) with $V \subseteq \omega$ and $Q \subseteq \omega \times \omega$ to which we associate the graph X_γ with vertex set V and edge set $Q \cap V \times V$. Every (countable) graph is isomorphic to one of this type. Let $\mathcal{X}$ be a class of graphs which is closed under isomorphism. A **mean** m on $\mathcal{X}$ is a map assigning to each $X \in \mathcal{X}$ a mean m_X on $\ell^\infty(\mathcal{V}(X))$, where $\mathcal{V}(X)$ denotes the vertex set of X. Such a mean is called **invariant** if for all isomorphisms $\pi : X \to Y \in \mathcal{X}$ and all $f \in \ell^\infty(\mathcal{V}(X))$, $m_X(f) = m_Y(f \circ \pi^{-1})$. The mean is called **universally measurable** if for every Borel set $B \subseteq \mathcal{G}$ such that $\gamma \in B \Rightarrow X_\gamma \in \mathcal{X}$, the map $F_B : \mathcal{G} \times {}^{\omega}[-1, 1] \to [-1, 1]$ given by

$$F_B((V, Q), f) := \begin{cases} m_{X_{(V,Q)}}(f \restriction V) & \text{if } (V, Q) \in B \\ 0 & \text{if } (V, Q) \notin B \end{cases}$$

is universally measurable.

Definition: A class $\mathcal{X}$ of graphs closed under isomorphism is **amenable** if there is a universally measurable invariant mean on $\mathcal{X}$.

The importance of amenable classes, as alluded to above, lies in the following implication [Kec, Proposition 2.6]:

PROPOSITION 2.5: *If (S, R, Q) is graphed equivalence relation, $\mathcal{X}$ is an amenable class of graphs, and each graph in (S, R, Q) belongs to $\mathcal{X}$, then (S, R) is amenable.*

3. Locally Finite Graphs

From now on, a **graph** for us will always be nonempty, countable, connected, and locally finite, except that subgraphs will be allowed to be disconnected when considering percolation. We give graphs the discrete topology and counting measure, card, implicitly. The automorphism group of a graph X is denoted Aut X and acts on the right. It is given the weak topology generated by the maps $g \mapsto xg$ ($x \in \mathcal{V}(X)$, $g \in \mathrm{Aut}X$), which is a locally compact Polish group topology. The most important feature of this topology and the local finiteness assumption is that the stabilizer of every vertex is compact. For a closed subgroup G of Aut X, we denote the stabilizer of $x \in X$ by G_x. This compactness entails amenability of the action:

PROPOSITION 3.1: *If X is a graph and G is a closed subgroup of* Aut$X$, *then the action of G on (X, card) is amenable. Hence G is amenable iff the action of G on (X, card) is co-amenable.*

Proof: The assertion on co-amenability is a consequence of Proposition 2.4, so it remains to establish amenability of the action. Let $(\alpha, \langle A_\bullet \rangle)$ be an affine space over X. Pick a transversal W of $\mathcal{V}(X)/G$. For each $v \in W$, pick $a_v \in A_v$ invariant under the compact group $\alpha(v, G_v)^*$. Define $\varphi : \mathcal{V}(X) \to E^*$ by

$$\varphi(vg) = \alpha(v, g)^* a_v \qquad (v \in W,\ g \in G).$$

In order to show that φ is well defined, note that

$$\alpha^*(v, g)\alpha^*(vg, g^{-1}) = \alpha^*(v, gg^{-1}) = \alpha^*(v, \mathrm{id}),$$

whence

$$\alpha^*(vg, g^{-1}) = \alpha(v, g)^* \alpha^*(v, \mathrm{id}).$$

Therefore, if $vg = vh$, we see that

$$\begin{aligned}
\alpha(v, h)^* a_v &= \alpha(v, h)^* \alpha^*(v, hg^{-1}) \alpha(v, hg^{-1})^* a_v \\
&= \alpha(v, h)^* \alpha^*(v, h) \alpha^*(vh, g^{-1}) a_v \\
&= \alpha^*(vg, g^{-1}) a_v = \alpha(v, g)^* \alpha^*(v, \mathrm{id}) a_v \\
&= \alpha(v, g)^* a_v,
\end{aligned}$$

having used that hg^{-1}, id $\in G_v$. Thus φ is well defined and, vacuously, Borel. Clearly φ is a section of $\langle A_\bullet \rangle$, so it remains to demonstrate its invariance. This stems from an easy calculation:

$$\alpha^*(vg,h)\varphi((vg)h) = \alpha^*(vg,h)\alpha(v,gh)^* a_v = \alpha(v,g)^* a_v = \varphi(vg).$$

□

A general result pointing more in the direction of amenability of groups, rather than amenability of actions, is that the growth of Aut X is dominated by that of X. The **distance**, $d(v,w)$, between $v, w \in \mathcal{V}(X)$ is defined to be the minimum number of edges in a path from v to w.

PROPOSITION 3.2: *If X is a graph and K is a compact neighborhood of the identity in* Aut$X$, *then*

$$\forall v \in \mathcal{V}(X)\ \exists c \quad \forall n \in \mathbf{N} \quad \rho_{\mathrm{Aut}X}(K^n) \le \rho_{\mathrm{Aut}X}((\mathrm{Aut}X)_v)\mathrm{card}\ B_v(cn),$$

where $B_v(r)$ denotes the ball of radius r centered at v.

Proof: Given $v \in \mathcal{V}(X)$, vK is compact, *i.e.*, finite, whence $c := \max_{g\in K} d(v,vg)$ is finite. If $g_1, \ldots, g_n \in K$, then since Aut X acts by isometries,

$$\begin{aligned} d(vg_1g_2\ldots g_n,\ v) &\le \sum_{k=1}^{n} d(vg_k\ldots g_n,\ vg_{k+1}\ldots g_n) \\ &= \sum_{k=1}^{n} d(vg_k,\ v) \le cn. \end{aligned}$$

Thus $vK^n \subseteq B_v(cn)$. For $w \in vK^n$, choose $g_w \in K^n$ so that $vg_w = w$. Then whenever $vg = w$, we have $g \in (\mathrm{Aut}X)_v g_w$. It follows that

$$\begin{aligned} \rho_{\mathrm{Aut}X}(K^n) &\le \sum_{w\in vK^n} \rho_{\mathrm{Aut}X}((\mathrm{Aut}X)_v g_w) = \sum_{w\in vK^n} \rho_{\mathrm{Aut}X}((\mathrm{Aut}X)_v) \\ &\le \rho_{\mathrm{Aut}X}((\mathrm{Aut}X)_v)\cdot \mathrm{card}\ B_v(cn). \end{aligned}$$

□

We say that X is of **subexponential growth** if $\liminf_{n\to\infty} \mathrm{card}\ B_v(n)^{1/n} = 1$ for some (hence every) $v \in \mathcal{V}(X)$. In this case, Aut X is also of subexponential growth by the above proposition, hence amenable. If X satisfies a stronger condition, then there is a "uniform" quality to this amenability:

PROPOSITION 3.3 [DK]: *Assume CH. The class $\mathcal{X}$ of graphs X such that*

$$\lim_{n\to\infty} \mathrm{card}\ B_v(n)^{1/n} = 1$$

for $v \in \mathcal{V}(X)$ is amenable.

For the proof, we require a theorem of Mokobodzki. A mean m on $\ell^\infty(V)$ is called **universally measurable** if $m \restriction {}^V[-1,1]$ is universally measurable.

MOKOBODZKI'S THEOREM [Mey], [DM, p. 113]: *Assume CH. There are universally measurable shift invariant means $m_{\mathbf{Z}}$ on $\ell^\infty(\mathbf{Z})$ and $m_{\mathbf{N}}$ on $\ell^\infty(\mathbf{N})$.*

It is not known whether this can be proved from ZFC alone. We will use $m_{\mathbf{Z}}$ and $m_{\mathbf{N}}$ throughout to denote such means, fixed once and for all.

Proof of Proposition 3.3: Given $X \in \mathcal{X}$ and $v \in \mathcal{V}(X)$, define $\Phi_v : \ell^\infty(\mathcal{V}(X)) \to \ell^\infty(\mathbf{N})$ by

$$\Phi_v(f)(n) := \frac{1}{\text{card } B_v(n)} \sum_{w \in B_v(n)} f(w)$$

and $C : \ell^\infty(\mathbf{N}) \to \ell^\infty(\mathbf{N})$ by

$$C(f)(n) := \frac{1}{n+1} \sum_{k=0}^{n} f(k).$$

Now set

$$m_X := m_{\mathbf{N}} \circ C \circ \Phi_v.$$

Given $v \in \mathcal{V}(X)$, card $B_v(n+1)$ card $B_v(n) \to 1$ as $n \to \infty$ along a set of density 1, whence for $w \in \mathcal{V}(X)$ and $f \in \ell^\infty(\mathcal{V}(X))$, $\Phi_v(f) - \Phi_w(f) \to 0$ along a set of density 1. Therefore

$$C(\Phi_v(f) - \Phi_w(f)) \to 0,$$

whence

$$m_{\mathbf{N}} \circ C \circ \Phi_v = m_{\mathbf{N}} \circ C \circ \Phi_w;$$

in other words, m_X is independent of choice of v. Thus, m_X is invariant. To show that it is universally measurable, note that if $(V,Q) \in \mathcal{G}$ with $X_{(V,Q)} \in \mathcal{X}$, then for $f \in {}^\omega[-1,1]$,

$$m_{X_{(V,Q)}}(f \restriction V) = m_{\mathbf{N}} \circ C \circ \Phi_{\min V}(f \restriction V).$$

Since $((V,Q), f) \mapsto \Phi_{\min V}(f \restriction V)$, C, and $m_{\mathbf{N}}$ are universally measurable, so is their composition. □

4. Trees

We shall often regard N and **Z** as trees with edges joining n to $n+1$ for all n. An **end** of a tree T is the equivalence class of an isometry $\pi : \mathbf{N} \to T$ under the equivalence relation $\pi \sim \pi' \Leftrightarrow \exists m, n\ \forall \ell \geq 0\ \pi(m+\ell) = \pi'(n+\ell)$. The space of ends is called the **boundary** of T and denoted ∂T. The union $T \cup \partial T$ is written $\overline{T}$. A topology is given on $\overline{T}$ which makes it compact. Namely, if $S \subseteq \mathcal{V}(T)$, write $\overline{S} := S \cup \{s \in \partial T;\ \exists \pi \in s\ \ \mathrm{im}(\pi) \subseteq S\}$. Then the neighborhood base for $s \in \partial T$ is

$$\{\overline{S};\ (\exists K \subseteq T \text{ finite}) \ \&\ (S \text{ is a component of } T \setminus K) \ \&\ (\exists \pi \in s\ \ \mathrm{im}(\pi) \subseteq S)\}.$$

A **line** in T is the image of an isometry from **Z** into T. Given distinct ends, s and t, there is a unique line "joining" s and t which we denote $\overleftrightarrow{st}$.

A remarkable theorem of Nebbia [N] characterizes amenable subgroups of automorphism groups of trees. We shall present a simplified proof of it here.

THEOREM 4.1 [N]: *Let T be a tree and G be a closed subgroup of* Aut T. *The following are equivalent:*

(i) *G is amenable;*
(ii) *G leaves invariant a set of cardinality 1 or 2 in $\overline{T}$*
(iii) *G leaves invariant a vertex, edge, end, or line.*

Proof: The group G acts affinely on the set of Borel probability measures on ∂T, which form a compact convex set in the weak* topology. By Day's Theorem, there is a fixed point, μ, if G is amenable. If card(supp μ) ≤ 2, then (ii) is evident. If not, consider the set $D := \{(s,t,u) \in (\partial T)^3;\ s \neq t \neq u \neq s\}$ and the map $p : D \to \mathcal{V}(T)$ given by $p(s,t,u) = \overleftrightarrow{st} \cap \overleftrightarrow{tu} \cap \overleftrightarrow{us}$. Since supp μ has at least three points, $\mu^3(D) > 0$ and so $\nu := (\mu^3 \restriction D) \circ p^{-1}$ is a nonzero G-invariant measure on $\mathcal{V}(T)$. Let

$$S := \{x \in \mathcal{V}(T);\ \nu(\{x\}) = \max_{y \in \mathcal{V}(T)} \nu(\{y\})\}.$$

The set S is finite and G-invariant. Its convex hull is a tree whose center is a G-invariant set of cardinality 1 or 2. Thus (i) $\Rightarrow$ (ii).

That (ii) $\Rightarrow$ (iii) is evident, so suppose that (iii) holds. If G leaves a vertex or an edge invariant, then G is compact, hence amenable. If an end s is fixed, then choose a shift-invariant mean m_0 on $\ell^\infty(\mathbf{N})$ and define $m \in \ell^\infty(\mathcal{V}(T))^*$ by $m(f) := m_0(f \circ \pi)$ for some (hence any) $\pi \in s$. Since s is fixed, m is G-invariant, whence the action of G on T is co-amenable. By Proposition 3.1, G is itself amenable. Similarly, if a line is invariant, say the image of $\pi : \mathbf{Z} \to T$, choose a shift-invariant mean m_0 on $\ell^\infty(\mathbf{Z})$ and define $m \in \ell^\infty(\mathcal{V}(T))^*$ by $m(f) := m_0(n \mapsto \frac{1}{2}(f(\pi(n)) + f(\pi(-n))))$. Then m is G-invariant, so G is amenable as before. □

We remark that Proposition 3.1 can be avoided in this proof by a direct lifting of the mean m to an invariant mean on $L^\infty(G)$ or by using Proposition 4.3 following. Before proceeding with further analyses of trees, we shall show how our proof of Nebbia's Theorem also gives a simple proof of an extension due to Woess [Woe] for graphs. In the context of graphs, an **infinite path** is a monomorphism $\pi : \mathbf{N} \to X$. Two infinite paths π, π' are **equivalent** if whenever K is a finite subset of $\mathcal{V}(X)$, there is a finite path in $T\backslash K$ connecting some vertex of $\mathrm{im}(\pi)$ to some vertex of $\mathrm{im}(\pi')$. An **end** is an equivalence class of infinite paths; the space of ends is denoted ∂X. The topology for $\overline{X} := X \cup \partial X$ is as it was for trees, which makes $\overline{X}$ compact.

There is an alternative definition of the space of ends of X. For any subset X_0 of the vertices X, let $H_0(X_0)$ denote the discrete topological space with one point for every connected component of the full subgraph of X with vertex set X_0. Then the space of ends of X may be identified with the inverse limit of $H_0(X\backslash F)$, as F ranges over finite subsets of X.

THEOREM 4.2 [Woe]: *Let X be a graph and G be a closed subgroup of* Aut$X$. *If G is amenable, then G leaves invariant a finite set in X or a set of cardinality 1 or 2 in ∂X.*

Proof: Let $D := \{(s,t,u) \in (\partial X)^3;\ s \neq t \neq u \neq s\}$. By definition, for $(s,t,u) \in D$, there is a finite set in $\mathcal{V}(X)$ which separates each pair of s,t,u. Let $K(s,t,u)$ be the collection of such finite sets which have minimum diameter. We claim that $K(s,t,u)$ is finite. If not, then fix $K \in K(s,t,u)$ and let C_s, C_t, C_u be the components of $X\backslash K$ containing $s, t,$ and u respectively. Since $\cup K(s,t,u)$ is infinite, it must intersect infinitely often some $\pi \in u$, say, with $\mathrm{im}(\pi) \subseteq C_u$. Choose $\pi_s \in s$, $\pi_t \in t$ with $\mathrm{im}(\pi_s)$, $\mathrm{im}(\pi_t) \subseteq X\backslash C_u$ and $\pi_s(0) = \pi_t(0)$. Every path joining $\mathrm{im}(\pi)$ to $\mathrm{im}(\pi_s) \cup \mathrm{im}(\pi_t)$ passes through K, whence $\cup K(s,t,u)$ has points which are arbitrarily far from $\mathrm{im}(\pi_s) \cup \mathrm{im}(\pi_t)$. Yet every set in $K(s,t,u)$ intersects $\mathrm{im}(\pi_s) \cup \mathrm{im}(\pi_t)$ since it separates s from t, whence $K(s,t,u)$ has sets of arbitrarily large diameter, a contradiction.

Suppose now that G is amenable and does not leave invariant any set of cardinality 1 or 2 in ∂X. Then as before, there is a G-invariant probability measure μ on ∂X with support larger than two points. Define the measure ν on

$\mathcal{V}(X)$ by

$$\nu(\{x\}) := \int_D \frac{1}{\operatorname{card} \cup K(s,t,u)} \mathbf{1}_{\cup K(s,t,u)}(x)\ d\mu^3(s,t,u).$$

This is a G-invariant subprobability measure and

$$S := \{x;\ \nu(\{x\}) = \max_y \nu(\{y\})\}$$

is a finite G-invariant set. □

Returning to trees, now, we note that Nebbia [N] has also shown that when G is not amenable, then G contains a discrete copy of the free group on two generators, $\mathbf{F}_2$. This is in sharp contrast to the following exceptionally nice behavior when G is amenable:

Proposition 4.3: *Let T be a tree and G be a closed subgroup of* Aut T. *If G is amenable but not compact, then either*

(i) *G leaves exactly one line invariant and is an extension of a compact normal subgroup by $\mathbf{Z}$ or by $\mathbf{Z}_2 * \mathbf{Z}_2$, or*

(ii) *G leaves exactly one end invariant, there is an increasing sequence $\langle K_n \rangle_{n \geq 1}$ of compact open subgroups of G such that each element of G belongs to the normalizer of all but finitely many K_n, and G is the union $\cup_n K_n$, possibly extended by $\mathbf{Z}$.*

Note that if $1 \to N \xrightarrow{i} G \xrightarrow{\theta} H \to 1$ is exact, N is compact, and H is $\mathbf{Z}$ or $\mathbf{Z}_2 * \mathbf{Z}_2$, then for every compact set $K \subseteq G$, there is a constant c such that

$$\forall n \geq 1 \quad \rho_G(K^n) \leq cn.$$

Indeed, if $F := \theta(K)$, then F is finite, so for some c, card $F^n \leq cn$, whence

$$\rho_G(K^n) \leq \rho_G(\theta^{-1}(F^n)) \leq \ \operatorname{card} F^n \cdot \rho_G(i(N)) \leq c\rho_G(i(N))n.$$

Similarly, if $G = \cup_n K_n$, K_n compact and open, then for compact $K \subseteq G$, $K \subseteq K_n$ for some n, whence $\rho_G(K^n)$ is bounded, while if $1 \to \cup K_n \to G \xrightarrow{\theta} \mathbf{Z} \to 1$ is exact with K_n as in (ii) above, then for compact $K \subseteq G$, $F := \theta(K)$ is finite, whence if $\theta(g)$ generates $\mathbf{Z}$, there is a $J \geq 1$ with $\theta(g^{[1,J]}) \supseteq F$. We have $K \subseteq g^{[1,J]} \cdot \cup K_n = \cup g^{[1,J]} K_n$ with $g^{[1,J]} K_n$ open, whence for all large n, $K \subseteq g^{[1,J]} K_n$. Also $gK_n = K_n g$ for all large n by hypothesis, whence for all $\ell \geq 1$, $K^\ell \subseteq g^{[1,J\ell]} K_n$. Again, we find that $\rho_G(K^\ell) \leq J\rho_G(K_n)\ell$.

Thus, closed amenable subgroups of Aut T have at most linear growth, while nonamenable ones grow exponentially. We also note that compact subgroups of Aut X for any graph, X, have a special form, namely, they are inverse limits along N of finite groups: Observe that if G is compact and $x \in \mathcal{V}(X)$, then xG is finite, so that $G_n := G \restriction (B_x(n)G)$ is a finite group. Clearly $G = \overleftarrow{\lim}\, G_n$.

Proof of Proposition 4.3: If G is amenable but not compact, then our proof of Nebbia's Theorem shows that G leaves invariant a set of cardinality 1 or 2 in ∂T, hence either a line or exactly one end, while not fixing any vertex or edge. Suppose that G leaves a line, $\pi(\mathbf{Z})$, invariant. Let $N := G_{\pi(0)} \cap G_{\pi(1)}$, $H := G \restriction \pi(\mathbf{Z})$, and $\theta : G \to H$ be restriction. Then $1 \to N \to G \xrightarrow{\theta} H \to 1$ is obviously exact. Now H is an infinite subgroup of Aut $(\pi(\mathbf{Z}))$ (otherwise G would be compact), whence H is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z}_2 * \mathbf{Z}_2$.

Now suppose that G fixes exactly one end $s \in \partial T$. For each $g \in G$, there is a unique $\theta(g) \in \mathbf{Z}$ such that for all $\pi \in s$ and all but finitely many k, $\pi(k)g = \pi(k + \theta(g))$. It is clear that θ is a homomorphism with kernel $\cup_n G_{\pi(n)}$ for any $\pi \in s$. Furthermore, the range of θ is isomorphic to either $\{0\}$ or $\mathbf{Z}$. Finally, for any $g \in G$ and $\pi \in s$, suppose that $\pi(k)g = \pi(k + \theta(g))$ for $k \geq k_0$. Then g normalizes $G_{\pi(n)}$ for all $n \geq k_0$. □

It is reasonable to expect that if Aut T is not amenable, then T cannot be too "small." Indeed, it is not hard to convince oneself that, since Aut T must then contain two (free) translations along two lines not sharing any end [N], T must contain a subtree which looks like a distorted version of a homogeneous tree of degree 3. In fact, this distorted homogeneous tree is the image under the free group generated by these two translations of the two lines together with the path connecting them. Thus, the set of distances from any given vertex of degree three in this distorted tree to its "neighbors" of degree three is always the same. In particular, T cannot have subexponential growth, as we knew from Proposition 3.3. A much broader notion of "small" tree is as follows. Given T and $0 \in \mathcal{V}(T)$, define a metric d on ∂T by setting $d(s,t) = e^{-n}$ if there are $\pi \in s$ and $\pi' \in t$ with $\pi(0) = \pi'(0) = 0$, $\pi(n) = \pi'(n)$, and $\pi(n+1) \neq \pi'(n+1)$. Computing the Hausdorff dimension in this metric, we define the **branching number** of T as [L1]

$$\text{br}\ T := e^{\dim\ \partial T}.$$

This measures the average rate of branching in several senses [L1, L2, LP, L3] and is independent of choice of root $0 \in \mathcal{V}(T)$. All trees of subexponential growth have branching number 1. Although trees of branching number 1 can have exponential growth, they cannot contain the kind of distorted homogeneous tree encountered above, whence they too have amenable automorphism groups. Nevertheless, this kind of reasoning does not show that a graphed equivalence space, almost all of whose graphs are trees of branching number one, is amenable. In order to prove this true statement, we require an argument utilizing percolation.

Percolation on a graph X with **survival parameter** $p \in [0,1]$ is the process of choosing a random subgraph $X(\omega)$ of X by keeping each edge independently with probability p. By Kolmogorov's $0-1$ law, the probability that $X(\omega)$ has an infinite connected component is either 0 or 1; we define the **critical probability** of X by

$$p_c(X) := \sup\{p \geq 0;\ \mathbf{P}_p[X(\omega) \text{ has an infinite component}] = 0\}.$$

We shall need the fact that for trees, $p_c(T) = 1/\operatorname{br} T$ [L1, L3]. Actually, we only require this result when $\operatorname{br} T = 1$, which is sufficiently simple to prove that we give it here. A translation of the definition [L1] gives that if $\operatorname{br} T = 1$, $p < 1$, and $\varepsilon > 0$, then there is a collection Π of vertices whose removal from T would leave 0 in a connected component of only finitely many vertices and such that

$$\sum_{x \in \Pi} p^{d(0,x)} < \varepsilon.$$

Therefore if $T(\omega)_0$ denotes the component of 0 in $T(\omega)$,

$$\begin{aligned}\mathbf{P}_p[\text{ card } T(\omega)_0 = \infty] \leq \mathbf{P}_p[T(\omega)_0 \cap \Pi \neq \emptyset] &\leq \sum_{x \in \Pi} \mathbf{P}_p[x \in T(\omega)_0] \\ &= \sum_{x \in \Pi} p^{d(0,x)} < \varepsilon,\end{aligned}$$

whence $T(\omega)_0$ is finite almost surely. Since the choice of root, 0, was arbitrary, all components are finite a.s., whence $p_c(T) = 1$, as desired.

Since the dimension of a denumerable set is 0, the following result implies [A, Theorem 5.2, p. 12] and [DK, Propositions 1 and 2].

THEOREM 4.4: *Assume CH. The class $\mathcal{X}$ of trees of branching number* 1 *is amenable.*

Proof: For $x \in \mathcal{V}(T)$, let $T(\omega)_x$ denote the component containing x of $T(\omega)$. Given $T \in \mathcal{X}$, $f \in \ell^\infty(\mathcal{V}(T))$, and $0 \leq p < 1$, set

$$F(p,f,x) := \mathbf{E}_p\left[\sum_{y \in T(\omega)_x} f(y)/\text{ card } T(\omega)_x\right].$$

(Here, card counts vertices.) Given $x, y \in T$, the probability that $T(\omega)_x = T(\omega)_y$ is the probability that x is connected to y in $T(\omega)$, namely, $p^{d(x,y)}$. Therefore

$$|F(p,f,x) - F(p,f,y)| \leq 2(1 - p^{d(x,y)})\|f\|_\infty. \tag{4.1}$$

In order to define a mean on $\mathcal{X}$, fix a sequence $\langle p_n \rangle \subseteq [0,1[$ converging to 1. For $T \in \mathcal{X}$, set

$$m_T(f) := m_{\mathbf{N}}(F(p_\bullet, f, x)) \quad (f \in \ell^\infty(\mathcal{V}(T))),$$

where the choice of $x \in \mathcal{V}(T)$ does not matter by (4.1). It is readily verified that $m_\bullet$ is universally measurable and invariant. □

As mentioned previously, this result encompasses Proposition 3.3 as far as trees are concerned. Of course, the same proof shows that the class of graphs of critical probability one is amenable. However, this does not encompass all graphs of subexponential growth, not even the square lattice on $\mathbf{Z}^2$, since $p_c(\mathbf{Z}^2) = 1/2$ [Gri]. It would be interesting to find a natural amenable class of graphs which includes both those of critical probability one and those of subexponential growth.

From Proposition 2.5 and metamathematical techniques, we may conclude the following.

COROLLARY 4.5: *Let (S, R, Q) be a graphed equivalence relation and μ be a Borel probability measure on S. If for μ-a.e. s, the graph associated to [s] is a tree of branching number one, then (S, μ, R) is amenable.*

It may be worthwhile to record a proof of this corollary which avoids metamathematics:

Alternative Proof: The proof consists of applying the basic idea of the proof of Theorem 4.4 in the context of affine spaces, weak* limits replacing $m_{\mathbf{N}}$. We shall keep the notation of the proof of Theorem 4.4. Let $(\alpha, \langle A_\bullet \rangle)$ be an affine space over (S, μ, R). Choose [Mos, p. 254] a Borel section f and define $(\alpha^* f)_s \in \ell^\infty([s])$ by

$$(\alpha^* f)_s(t) := \alpha^*(s,t) f(t).$$

For $p \in [0,1[$, define $\varphi_p : S \to E^*$ by

$$\langle e, \varphi_p(s) \rangle := F(p, \langle e, (\alpha^* f)_s \rangle, s) \qquad (e \in E).$$

By the Hahn–Banach Theorem, φ_p is a section of $\langle A_\bullet \rangle$. For $(s,t) \in Q$, we have

$$\alpha^*(s,t)(\alpha^* f)_t = (\alpha^* f)_s,$$

whence

$$\langle \alpha(s,t)^{-1}e,\ (\alpha^* f)_t\rangle = \langle e,\ (\alpha^* f)_s\rangle,$$

and so

$$\begin{aligned}|\langle e, \varphi_p(s)\rangle - \langle e,\ \alpha^*(s,t)\varphi_p(t)\rangle| &= |\langle e, \varphi_p(s)\rangle - \langle \alpha(s,t)^{-1}e,\ \varphi_p(t)\rangle| \\ &= |F(p,\ \langle e,\ (\alpha^* f)_s\rangle, s) - F(p,\ \langle e,\ (\alpha^* f)_s\rangle,\ t)| \\ &\le 2(1-p^{d(s,t)})\end{aligned}$$

by (4.1). In other words,

$$\forall (s,t) \in R \quad \varphi_p(s) - \alpha^*(s,t)\varphi_p(t) \xrightarrow{w^*} 0 \text{ as } p \to 1. \tag{4.2}$$

Now the subset of sections in $L^\infty(S,\mu;E^*)$ is compact and metrizable in the weak* topology [Z2, Proposition 2.2], whence there is a sequence $p_n \to 1$ and a section φ such that

$$\varphi_{p_n} \xrightarrow{w^*} \varphi \quad \text{as } n \to \infty.$$

We claim that φ is α-invariant, i.e.,

$$\text{for } \mu\text{-a.e. } t \quad \forall s \in [t] \quad \varphi(s) = \alpha^*(s,t)\varphi(t). \tag{4.3}$$

To this end, recall the theorem of Feldman and Moore [FM, Theorem 1] that there is a countable group G of Borel automorphisms of S which generates R. The measure μ is automatically quasi-invariant under G. To prove (4.3), it suffices to establish that

$$\forall g \in G \quad \text{for } \mu\text{-a.e. } s \in S \quad \varphi(s) = \alpha^*(s,gs)\varphi(gs). \tag{4.3$'$}$$

By quasi-invariance of μ, we have, for every $g \in G$,

$$\varphi_{p_n} \circ g \xrightarrow{w^*} \varphi \circ g \quad \text{as } n \to \infty.$$

Now to show (4.3)$'$, consider any $f \in L^1(S,\mu;E)$. We have

$$\begin{aligned}0 &= \lim_{n\to\infty} \int_S \{\langle f(s),\ \varphi_{p_n}(s)\rangle - \langle \alpha(s,gs)^{-1} f(s,gs),\ \varphi_{p_n}(gs)\rangle\} d\mu(s) \\ &= \int_S \{\langle f(s),\ \varphi(s)\rangle - \langle \alpha(s,gs)^{-1} f(s,gs),\ \varphi(gs)\rangle\} d\mu(s),\end{aligned}$$

whence (4.3)$'$ follows. □

We would be very interested to see an analogue of this result for foliations by Cartan–Hadamard manifolds.

Theorem 4.4 has another interesting and, at first sight, surprising consequence. We shall say that a tree T' is obtained from **stretching** a tree T if T' is the result of subdividing the edges of T by adding more vertices to T (each of the new vertices therefore has degree 2). If T has an amenable automorphism group, then it is trivial by Nebbia's Theorem to stretch T in an Aut T-invariant way to obtain a tree of branching number one – in fact, of arbitrarily slow growth. (For the case where Aut T fixes exactly one end, note that Aut T cannot contain any translations along any line joining the fixed end to another end since it would then contain an automorphism switching the two ends of the line.) However, it is impossible to stretch T this much, even in a random way, if Aut T is not amenable:

COROLLARY 4.6: *If T is a tree whose automorphism group is not amenable, then there is no automorphism-invariant random stretching of T which yields a tree of branching number one with positive probability. In other words, if Q is the edge set of T, $\mathbf{P}$ is an* Aut T*-invariant Borel probability on ${}^Q\mathbf{N}$, $T(\omega)$ is the tree obtained from T by adding $\omega(q)$ vertices to q ($q \in Q$, $\omega \in {}^Q\mathbf{N}$), and* Aut T *is not amenable, then* br $T(\omega) > 1$ $\mathbf{P}$-a.s.

Proof: We shall prove this with the class of trees of branching number one replaced by any amenable class $\mathcal{X}$ of trees whose "intersection" with the parameter space $\mathcal{G}$ of Section 2 is universally measurable. (The use of CH in applying Theorem 4.4 is eliminated by metamathematics. Alternatively, one may avoid metamathematics and Mokobodzki's Theorem by rearranging the proof which follows; the implicit use of $m_{\mathbf{N}}$ in (4.4) coming from the proof of Theorem 4.4 is replaced by a Banach limit outside the expectation.) We shall prove that if $\mathbf{P}[T(\omega) \in \mathcal{X}] > 0$, then the action of Aut T on $\mathcal{V}(T)$ is amenable, which is sufficient by Proposition 3.1. Define $F(\omega) : \ell^\infty(\mathcal{V}(T)) \to \ell^\infty(\mathcal{V}(T(\omega)))$ as follows. If the edge $q = (x, y)$ is stretched to the path $x = x_0, x_1, \ldots, x_{\omega(q)}, x_{\omega(q)+1} = y$, set

$$F(\omega)(f)(x_i) := \frac{(\omega(q) + 1 - i)f(x) + if(y)}{\omega(q) + 1} \quad (f \in \ell^\infty(\mathcal{V}(T))).$$

This allows us to define $m \in \ell^\infty(\mathcal{V}(T))^*$ by

$$(4.4) \qquad m(f) := \mathbf{E}[m_{T(\omega)}(F(\omega)(f)) | T(\omega) \in \mathcal{X}],$$

where $m_\bullet$ is a universally measurable invariant mean on $\mathcal{X}$. It is readily verified that m is an Aut T-invariant mean. □

Remark: If T is stretched by adding vertices in an i.i.d. fashion on the edges (i.e., $\mathbf{P}$ is a product measure on ${}^{Q}\mathbf{N}$ with every one-dimensional marginal the same), then we may actually calculate br $T(\omega)$ a.s. By the $0-1$ law, it is constant a.s., and by considering percolation on $T(\omega)$ (cf. the proof of [L1, Proposition 6.1 or 6.4]), we see that br $T(\omega) = \lambda$ a.s., where $\lambda > 1$ is the solution of

$$\sum_{n\geq 1} p_n \lambda^{-n} = \frac{1}{\text{br } T},$$

where p_n is the probability of adding $n-1$ vertices to an edge.

We shall now examine two other classes of trees to see whether they are amenable. Of course, any amenable class can contain only trees with amenable automorphism groups. Surprisingly, we find that the bigger the group, the more amenable the class, to speak loosely. More specifically:

THEOREM 4.7: *The class $\mathcal{X}_1$ of trees with noncompact amenable automorphism group is amenable assuming CH, but the class $\mathcal{X}_2$ of rigid trees (i.e., those with trivial automorphism group) is not amenable.*

Proof: By virtue of Proposition 4.3, for each $T \in \mathcal{X}_1$, there is a unique end or line of T, call it $h(T)$, which is Aut T-invariant. In order to show that h is Borel in a suitable sense, we define the following parameter spaces. Let $\mathcal{E} :={}^{\omega}\omega$ parametrize ends with

$$\mathcal{E}(V,Q) : = \{\pi \in \mathcal{E};\ \forall n \in \omega \quad \pi(n) \in V \ \&\ (\pi(n), \pi(n+1)) \in Q$$
$$\&\forall m \in \omega \quad n \neq m \Rightarrow \pi(n) \neq \pi(m)\}$$

for $(V,Q) \in \mathcal{G}$, let $\mathcal{L} :={}^{\mathbf{Z}}\omega$ parametrize lines with

$$\mathcal{L}(V,Q) : = \{\pi \in \mathcal{L};\ \forall n \in \mathbf{Z}\ \pi(n) \in V\ (\pi(n), \pi(n+1)) \in Q$$
$$\&\forall m \in \omega \quad n \neq m \Rightarrow \pi(n) \neq \pi(m)\},$$

and $\mathcal{A} :={}^{\omega}\omega$ parametrize automorphisms, with

$$\mathcal{A}(V,Q) := \{g \in \mathcal{A};\ g \restriction V : V \rightarrowtail V \ \&\ (g,g) \restriction Q : Q \rightarrowtail Q\}.$$

Let $B \subseteq \mathcal{G}$ be Borel with $\gamma \in B \Rightarrow X_\gamma \in \mathcal{X}_1$. With slight abuse of notation, $h \restriction B : B \to \mathcal{E} \cup \mathcal{L}$ and, by uniqueness,

$$h(\gamma) = \pi \Leftrightarrow [\pi \in \mathcal{E}(\gamma)\ \forall g \in \mathcal{A}(\gamma)\ \ \text{card}(\text{ im } \pi \triangle \text{ im } g \circ \pi) < \infty]$$
$$\vee [\pi \in \mathcal{L}(\gamma)\ \&\ \forall g \in \mathcal{A}(\gamma)\ \ \text{im } \pi = \text{ im } g \circ \pi].$$

Now $\mathcal{A}(\gamma)$ is not only closed in $\mathcal{A}$, but σ-compact since

$$\mathcal{A}(V,Q) = \cup_{n\in\omega}\{g \in \mathcal{A}(V,Q);\ g(\min\ V) = n\},$$

whence $\{g \in \mathcal{A}(\gamma);\ \text{card}(\ \text{im}\ \pi\triangle\ \text{im}\ g\circ\pi) = \infty\}$, being open in $\mathcal{A}(\gamma)$, is also σ-compact in $\mathcal{A}$. Therefore

$$\{(\gamma,\pi,g) \in \mathcal{G}\times\mathcal{E}\times\mathcal{A};\ \pi\in\mathcal{E}(\gamma)\ \&\ g\in\mathcal{A}(\gamma)\ \&\ \text{card}(\ \text{im}\ \pi\ \triangle\ \text{im}\ g\circ\pi) = \infty\}$$

is Borel with σ-compact sections over $\mathcal{G}\times\mathcal{E}$, whence [Mos, 4F.16 and 2E.7]

$$\{(\gamma,\pi) \in \mathcal{G}\times\mathcal{E};\ \pi\in\mathcal{E}(\gamma)\ \&\ \exists g\in\mathcal{A}(\gamma)\ \ \text{card}(\ \text{im}\ \pi\ \triangle\ \text{im}\ g\circ\pi) = \infty\}$$

is Borel. Thus

$$\{(\gamma,\pi) \in \mathcal{G}\times\mathcal{E};\ \pi\in\mathcal{E}(\gamma)\ \&\ \forall g\in\mathcal{A}(\gamma)\ \ \text{card}(\ \text{im}\ \pi\ \triangle\ \text{im}\ g\circ\pi) < \infty\}$$

is also Borel. Similarly

$$\{(\gamma,\pi) \in \mathcal{G}\times\mathcal{E};\ \pi\in\mathcal{L}(\gamma)\ \&\ \forall g\in\mathcal{A}(\gamma)\ \ \text{im}\ \pi = \ \text{im}\ g\circ\pi\}$$

is Borel. Therefore $h \restriction B$ has a Borel graph, whence [Mos, 2E.4] h is Borel, as desired. By using $m_{\mathbf{N}}$ if $h(T)$ is an end and $m_{\mathbf{Z}}$ if $h(T)$ is a line, it is easy to construct a universally measurable invariant mean on $\mathcal{X}_1$ (cf. the proof of Theorem 4.1).

On the other hand, let T be a homogeneous tree of degree 4 (say). Since it contains a discrete copy of $\mathbf{F}_2$ (T being the Cayley graph of $\mathbf{F}_2$), Aut T is not amenable. Consider the random stretching of T which adds, independently to each edge, 0 or 1 vertices with equal probability. In view of the proof of Corollary 4.6, it suffices to show that the stretched trees $T(\omega)$ are a.s. rigid. Let $\mathcal{T}$ denote the set of subtrees of T all of whose vertices have degree 4 except one, which has degree 3. Any nontrivial automorphism of T moves some line to a distinct line and hence establishes an isomorphism between some pair of disjoint elements of $\mathcal{T}$. Furthermore, for any ω, every nontrivial automorphism of $T(\omega)$ induces a nontrivial automorphism of T. Since $\mathcal{T}$ is countable, it suffices to show that for disjoint $T_1, T_2 \in \mathcal{T}, T_1(\omega)$ and $T_2(\omega)$ are a.s. non-isomorphic (where $T_i(\omega)$ are the induced subtrees of $T(\omega)$). Let x_i be the vertex of T_i of degree 3 and y_i^j be the vertices in T_i adjacent to x_i ($i = 1,2,\ j = 1,2,3$). Let p be the probability that $T_1(\omega)$ and $T_2(\omega)$ are isomorphic. Any isomorphism must carry x_1 to x_2 and $\{y_1^j\}$ to $\{y_2^j\}$. There are 6 possible bijections of $\{y_1^j\}$ with $\{y_2^j\}$ and any isomorphism of $T_1(\omega)$ with $T_2(\omega)$ which induces a given one of these bijections must match edges (x_i, y_i^j) with the same stretching, which has probability $1/8$, and must also induce isomorphisms of the subtrees following y_i^j, which has probability p^3. Hence $p \le 6p^3/8$, whence $p = 0$. □

5. Actions of Fundamental Groups on Trees

Let G be a connected, noncompact, almost simple Lie group with finite fundamental group. Let M be a closed, real analytic manifold with a real analytic connection and a smooth volume. Assume that G acts real analytically on M preserving the connection and volume. Assume that the action is nontrivial. Let Γ denote the fundamental group of M.

Recall that the **R-rank** of G is the dimension of a maximal **R**-split torus in the algebraic **R**-group $\mathrm{Ad}(G)$. The group G is said to have **Kazhdan's property (T)** if any unitary representation which almost has invariant unit vectors actually has invariant unit vectors [Z3, Definition 7.1.3, p. 130]. It is known that **R**-rank ≥ 2 implies Kazhdan's property (T).

In [Gro, Theorem 6.1.B], Gromov proves that the universal covering action of $\tilde{G}$ on $\tilde{M}$ is proper (in the sense of measure theory). This result has been used by Zimmer to prove many results, among them:

THEOREM 5.1 [Z4, Theorem 7.1, p. 211]: *Assume that G has Kazhdan's property (T). Then Γ is not isomorphic to the amalgam of two finite groups over a common subgroup.*

In particular, $\Gamma \neq \mathrm{SL}(2, \mathbf{Z})$, because $\mathrm{SL}(2, \mathbf{Z})$ is the amalgam of $\mathbf{Z}/4\mathbf{Z}$ with $\mathbf{Z}/6\mathbf{Z}$ over $\mathbf{Z}/2\mathbf{Z}$.

Now, a continuous action of a topological group G on a topological space V is said to be **proper** if for every compact set $V_0 \subseteq V$, there exists a compact $K \subseteq G$ such that, for every $g \in G \backslash K$, we have $gV_0 \cap V_0 = \emptyset$. Here we show how the ideas in Zimmer's proof of the above result can be modified to show that Γ cannot act properly on a tree:

THEOREM 5.2: *Assume that G has Kazhdan's property (T). Let Γ act on a tree T by tree automorphisms. Give the vertices V of T the discrete topology. Then the action of Γ on V is not proper.*

This result generalizes Theorem 5.1 by virtue of [S, §4, Theorem 7, p. 32]. It implies that Γ is not isomorphic to a discrete group of the automorphism group of a tree.

A group Γ is said to have **Serre's property (FA)** if whenever Γ acts on a tree, it fixes a vertex or an edge [S, p. 58, l.+12]. Watatani [Wat] has shown that Kazhdan's property (T) implies Serre's property (FA). Under the assumptions of Theorem 5.2, the group Γ need not have property (FA), since one may take any example of G and M, then replace M by $M \times S^1$, letting G act trivially on the circle S^1.

On the other hand, if one imposes the condition that the action of G on M be measure-theoretically engaging [Z4, Definition 3.1, p. 205], we know of no example where Γ does not have property (FA). However, even in this case, the group need not have Kazhdan's property (T); see [Z4, p. 210, l.+4 to l.+11] and cf. [Z4, Proposition 6.1, p. 210].

Note that a group without proper tree actions does not necessarily have Serre's property (FA): Let Λ_1 be any infinite group with property (FA) and let Λ_2 be any nontrivial group. Then $\Lambda_1 * \Lambda_2$ does not have Serre's property (FA) and has no proper tree actions.

Theorem 5.2 is analogous to a result of Spatzier and Zimmer:

THEOREM 5.3 [SZ, Theorem A]: *Assume that G has $\mathbf{R}$-rank ≥ 2. If $0 < a \leq b$, then Γ cannot act properly on a simply connected complete Riemannian manifold with sectional curvature satisfying $-b^2 \leq K \leq -a^2$. In particular, M cannot carry a metric of negative curvature.* □

This theorem has some overlap with Theorem 5.2. For example, when G has $\mathbf{R}$-rank ≥ 2, they both imply that Γ cannot be isomorphic to $\mathrm{SL}(2, \mathbf{Z})$.

In Theorem 5.3, it is not possible to weaken the assumption of $\mathbf{R}$-rank ≥ 2 to Kazhdan's property (T): If $G = \mathrm{Sp}(1, n)$, Γ is a cocompact lattice in G and $M = G/\Gamma$, then Γ acts properly on quaternionic hyperbolic space, which has curvature bounds $-1 \leq K \leq -1/4$.

Spatzier and Zimmer's proof is quite involved. Using an ergodic-theoretic analogue [AS, Theorem 2.3, p. 276] of Watatani's theorem, we obtain a relatively simple proof of Theorem 5.2.

Our method works whenever Γ acts (properly) on a space with a proper, weakly negative semi-definite kernel and a barycenter process. In particular, by [FH, Proposition II.7.3, p. 205], we can recover the result of Spatzier and Zimmer in the special case where $0 < a = b$, i.e., a simply connected manifold of constant negative curvature. Similarly, by [FH, §II.7.4, p. 211], we see that Γ cannot act properly on complex hyperbolic space. We can also prove that Γ cannot act properly and isometrically on a flat Euclidean space. However, our result only requires that the group acting have Kazhdan's property (e.g., $\mathrm{Sp}\,(1, n), n \geq 2$, and F_4^{-20}), not that it have higher (real) rank. We summarize some of these remarks as follows:

THEOREM 5.4: *Assume that G has Kazhdan's property (T). Then the fundamental group of M is not isomorphic to a discrete subgroup of $\mathrm{SO}(1,n)$ or of $\mathrm{SU}(1,n)$ for any $n \geq 1$. In particular, M does not admit a real hyperbolic or complex hyperbolic metric.* □

Zimmer has noted that this result may also be proved using [Z5, Theorem 10, p. 427], [Gro, Theorem 6.1.B] and elementary facts about cocycles.

In Theorem 5.2, we are taking "tree" to mean "countable, simplicial tree". (We need *not* assume that the tree is locally finite.) However, since an $\mathbf{R}$-tree has a negative definite kernel (given by distance) and a barycenter process, Theorem 5.2 is true for $\mathbf{R}$-trees as well.

Proof of Theorem 5.2: Assume for a contradiction that Γ acts properly on the discrete topological space V.

In [Gro, Theorem 6.1.B], Gromov shows that the action of $\tilde{G}$ on $\tilde{M}$ is "proper in the sense of measure theory", i.e., there is a measurable map $\tilde{M} \to [0,1]$ such that a.e. fiber is a $\tilde{G}$-orbit with compact stabilizer. In particular, by disintegrating the smooth measure on $\tilde{M}$, we see that any $\tilde{G}$-invariant set in $\tilde{M}$ must have measure 0 or ∞.

Let $\pi : \tilde{G} \to G$ denote the natural map. Let $\tilde{G}$ act on M by $\tilde{g}m = \pi(\tilde{g})m$.

Fix a measurable fundamental domain for the action of Γ on $\tilde{M}$. This induces a measurable trivialization of $\tilde{M} \to M$ as the projection map $M \times \Gamma \to M$. The resulting $\tilde{G}$-action on $M \times \Gamma$ is given by $\tilde{g}(m,\gamma) = (\tilde{g}m, \alpha(\tilde{g},m)\gamma)$, for some cocycle $\alpha : \tilde{G} \times M \to \Gamma$.

The group $\tilde{G}$ acts on the associated bundle $\tilde{M} \times_\Gamma V \to M$, and the measurable Γ-equivariant bijection of $\tilde{M}$ onto $M \times \Gamma$ induces an identification of $\tilde{M} \times_\Gamma V$ with $M \times V$. Then $\tilde{G}$ acts on $M \times V$ via $\tilde{g}(m,v) = (\tilde{g}m, \alpha(\tilde{g},m)v)$.

By [AS, Theorem 2.3, p. 276], there exists a measurable α-invariant field of non-empty convex bounded subsets of V parametrized by M. Applying the barycenter (or "pruning") process to all of these convex bounded subsets, we arrive at a $\tilde{G}$-invariant measurable subset $F \subseteq M \times V$ such that a.e. fiber of $F \to M$ has one or two elements.

We think of $M \times \Gamma$ and of $M \times V$ as measure spaces, where M has its smooth volume and where Γ and V are given counting measure. Then the measurable identification of $\tilde{M}$ with $M \times \Gamma$ is measure-preserving. Further, since a.e. fiber of $F \to M$ has one or two elements, we conclude that F has positive, finite measure.

Choose $v_0 \in V$ such that $F_0 := F \cap (M \times \Gamma v_0)$ has positive measure. Let $\phi : \tilde{M} \to \tilde{M} \times_\Gamma V$ denote the composite of the map $m \mapsto (m, v_0) : \tilde{M} \to \tilde{M} \times V$

followed by the natural map $\tilde{M} \times V \to \tilde{M} \times_\Gamma V$. The identifications of $\tilde{M}$ with $M \times \Gamma$ and of $\tilde{M} \times_\Gamma V$ with $M \times V$ allow us to identify ϕ with a map $\psi : M \times \Gamma \to M \times V$. This map is given by $\psi(m, \gamma) = (m, \gamma v_0)$.

As ϕ is $\tilde{G}$-equivariant, so is ψ. Consequently, $M \times \Gamma v_0 = \psi(M \times \Gamma)$ is $\tilde{G}$-invariant. Then F_0 is the intersection of two $\tilde{G}$-invariant sets; it is therefore $\tilde{G}$-invariant. It is also of positive, finite measure.

Let n denote the cardinality of the stabilizer in Γ of v_0. Since Γ is proper on V, it follows that $n < \infty$. The map $\psi : M \times \Gamma \to M \times \Gamma v_0$ is n-to-1 and behaves well with respect to measure in the following sense: if a subset $S \subseteq M \times \Gamma v_0$ has measure a, then $\psi^{-1}(S)$ has measure na. Consequently, $\psi^{-1}(F_0)$ has positive, finite measure. By $\tilde{G}$-equivariance of ψ, we find that $\psi^{-1}(F_0)$ is also $\tilde{G}$-invariant.

Finally, the identification of $\tilde{M}$ with $M \times \Gamma$ is measure-preserving and $\tilde{G}$-equivariant. Thus $\psi^{-1}(F_0)$ corresponds to a $\tilde{G}$-invariant subset in $\tilde{M}$ with positive, finite measure. This contradicts Gromov's theorem. □

Appendix 1

Here we show the equivalence of the two definitions of amenable equivalence spaces given in Section 2.

Suppose, to start, that we are given an amenable equivalence space (S, μ, R) in the first sense (not Zimmer's). For any affine space $(\alpha, \langle A_\bullet \rangle)$ over (S, μ, R), choose (by [Mos, p. 254]) a Borel section g and a μ-measurable mean m on S/R. For $e \in E$, the function $F(e) : R \to \mathbf{R}$ defined by $F(e)(s,t) := \langle e, \alpha^*(s,t) g(t) \rangle$ is Borel, whence the map $\varphi : S \to E_1^*$ defined by

$$\langle e, \varphi(s) \rangle := m_{[s]}(F(e)_s)$$

is μ-measurable. In view of the Hahn–Banach Theorem, φ is a section of $\langle A_\bullet \rangle$. It remains to show that φ is α-invariant. Now for $(s,t), (t,u) \in R$ and $e \in E$, we have

$$\begin{aligned} F(\alpha(s,t)^{-1} e)(t,u) &= \langle \alpha(s,t)^{-1} e,\ \alpha^*(t,u) g(u) \rangle \\ &= \langle e, \alpha^*(s,t)\alpha^*(t,u) g(u) \rangle = \langle e, \alpha^*(s,u) g(u) \rangle \\ &= F(e)(s,u), \end{aligned}$$

whence

$$F(\alpha(s,t)^{-1} e)_t = F(e)_s.$$

Since $[s] = [t]$, we obtain

$$\begin{aligned} \langle e, \alpha^*(s,t)\varphi(t) \rangle &= \langle \alpha(s,t)^{-1} e,\ \varphi(t) \rangle = m_{[t]}(F(\alpha(s,t)^{-1} e)_t) = m_{[s]}(F(e)_s) \\ &= \langle e, \varphi(s) \rangle, \end{aligned}$$

which is to say, φ is α-invariant. Thus, the equivalence space is amenable in Zimmer's sense.

Conversely, let (S,μ,R) be an equivalence space which is amenable in the sense of Zimmer. Our first step is to show that there exists a "global" right-invariant mean on R in the sense of [CFW, Definition 5, p. 437]. This is done in [Z1, Proposition 4.1 (ii), p. 30]. In part (i) of that proposition, an ergodicity assumption is made. In fact, in [Z1], amenability is only defined for ergodic actions and ergodic equivalence relations. However, ergodicity is never used. The proof of [Z1, Proposition 4.1] proceeds via hyperfiniteness of the von Neumann algebra of the action or equivalence relation. We shall sketch a direct proof below.

The second step is to apply the main result of [CFW] to conclude that there exists a conull, Borel, R-invariant subset $S_0 \subseteq S$ and a $\mathbf{Z}$-action on S_0 such that the equivalence classes of $R_0 := R \restriction S_0$ are exactly the orbits of this $\mathbf{Z}$-action. Let $\mu_0 := \mu \restriction S_0$. Since $\mathbf{Z}$ is an amenable group, it follows that there is a μ_0-measurable mean on S_0/R_0. Since S_0 is conull, any extension of this mean to S/R will be μ-invariant.

We now describe the direct proof of the first step.

PROPOSITION: *If (S,μ,R) is amenable in the sense of Zimmer, then there is a right-invariant mean on R in the sense of [CFW, Definition 5, p. 437].*

Sketch of Proof: By [FM, Theorem 1, p. 291], there is a countable group G of Borel automorphisms of S such that the equivalence classes of R are the orbits of G. By ordering G, we may choose a Borel map $\Phi : R \to \mathbf{N}$ such that, for every $s \in S$, the map $\Phi_s : [s] \to \mathbf{N}$ given by $\Phi_s(s') := \Phi(s,s')$ is bijective. Define $\alpha : R \to \mathrm{Perm}(\mathbf{N})$, where $\mathrm{Perm}(\mathbf{N})$ denotes the **permutation group** of $\mathbf{N}$, by $\alpha(s,s') := \Phi_s \circ \Phi_{s'}^{-1}$.

Put the product of μ and counting measure, $\mu \times$ card, on $S \times \mathbf{N}$. If $A \in L^\infty(S \times \mathbf{N})$, then we will say that A is **α-invariant** if, for μ-a.e. $s \in S$, for all $s' \in [s]$, we have $A(s,n) = A(s',\alpha(s',s)n)$.

For every $A \in L^\infty(S\times\mathbf{N})$, for every $g \in G$, let $T_g(A) \in L^\infty(S\times\mathbf{N})$ be defined by $T_g(A)(s,n) = A(g^{-1}s, \alpha(g^{-1}s,s)n)$. For every $A \in L^\infty(S\times\mathbf{N})$, for every $f \in L^\infty(S)$, let $U_f(A) \in L^\infty(S\times\mathbf{N})$ be defined by $U_f(A)(s,n) = f(s)A(s,n)$.

Give $B := B(L^\infty(S\times\mathbf{N}))$ the $\sigma(B, L^\infty(S\times\mathbf{N})\otimes_{\max}L^1(S\times\mathbf{N}))$ topology, a weak* topology. Let X denote the closure in B of the set of operators of the form $\sum_{i=1}^n T_{g_i}\circ U_{f_i}$, where $g_1,\dots,g_n \in G$ and $f_1,\dots,f_n \in L^\infty(S)$ are nonnegative and satisfy $f_1+\cdots+f_n = \mathbf{1}$. Then X is a semigroup, by the argument of [Z1, p. 27, l.+12 to l.+18]. Give X the inherited topology from B; this makes X

compact. The elements of X are "left rearrangement operators" in the sense that each one takes a function on $S \times \mathrm{N}$, breaks it down by a partition of unity from S, moves the pieces around (using elements of G and the cocycle α), then reassembles the pieces.

Since R is amenable, we may mimic the argument in [Z1, Lemma 2.3, pp. 25–26] to conclude that for every $A \in L^\infty(S \times \mathrm{N})$, $XA := \{VA; V \in X\}$ contains an α-invariant element. ([Z1] uses $C_S(A)$ to denote what is here denoted XA; it is just the set of "left rearrangements of A".)

Note that if $A \in L^\infty(S \times \mathrm{N})$ is α-invariant, then $XA = \{A\}$. For $A \in L^\infty(S \times \mathrm{N})$, write

$$X_A := \{V \in X;\ VA \quad \text{is} \quad \alpha\text{-invariant}\}.$$

Then X_A is non-empty and closed. Furthermore, if $A_1, \dots, A_n \in L^\infty(S \times \mathrm{N})$, choose $V_1 \in X_{A_1}$ and, for $k = 2, \dots, n$, $V_k \in X_{V_{k-1}\cdots V_1 A_k}$. Then

$$V_n V_{n-1} \cdots V_1 \in \bigcap_{i=1}^{n} X_{A_i},$$

whence $\{X_A\}$ has the finite intersection property. Therefore

$$\exists W \in \bigcap_{A \in L^\infty(S \times \mathrm{N})} X_A.$$

Fix any $A \in L^\infty(S \times \mathrm{N})$. Then WA is α-invariant. It follows that, for μ-a.e. $s \in S$, for every $s', s'' \in [s]$, we have

$$(WA)(s', \Phi(s', s)) = (WA)(s'', \Phi(s'', s)).$$

Give R the measure class described in [CFW, p. 434, l.+12 to l.+19]. Fix $a \in L^\infty(R)$. Define $A \in L^\infty(S \times \mathrm{N})$ by $A(s,n) := a(s, \Phi_s^{-1}(n))$. Define $\bar{a} \in L^\infty(S)$ by $\bar{a}(s) := (WA)(s, \Phi(s,s))$. Then $a \mapsto \bar{a} : L^\infty(R) \to L^\infty(S)$ is a right-invariant mean in the sense of [CFW, Definition 5, p. 437]. □

Appendix 2

We shall outline how metamathematics can be used to eliminate the use of CH in statements about standard measure spaces. We shall restrict ourselves to theorems whose conclusion is that (S, μ, R) is amenable, which should suffice for illustrative purposes. By [R, Proposition 15.12, p. 407], we may assume that $S = [0,\ 1]$. We suppose that the hypotheses form a projective formula, i.e., that they can be expressed using quantifiers over $[0,\ 1]$ but not over the power set of $[0,\ 1]$ or any higher order object. Then it suffices to show that "$([0,\ 1], \mu, R)$ is amenable" is also projective. This is because of the following folklore theorem.

THEOREM: *If θ is a projective sentence and is a theorem of ZFC+CH, then θ is a theorem of ZFC.*

Proof: It suffices to show that θ is true, *i.e.*, is satisfied in any model, $\mathcal{M}$, of ZFC. The method of forcing gives an extension, $\mathcal{M}'$, of $\mathcal{M}$ in which CH is satisfied but which has the same set for $[0,\ 1]$. Since θ is satisfied in $\mathcal{M}'$ by hypothesis and θ is projective, it follows that θ is satisfied in $\mathcal{M}$. □

To show that "$([0,\ 1], \mu, R)$ is amenable" is projective, we first demonstrate a well-known universal embedding of separable Banach spaces.

PROPOSITION: *If E is a separable Banach space, then there is an isometric isomorphism of E onto a closed subspace of $C({}^{\omega}\{0,\ 1\})$.*

Proof: Since E is separable, the topology of E_1^* is metrizable and, of course, compact. Therefore, there is a continuous surjection $\pi : {}^{\omega}\{0,\ 1\} \to E_1^*$. The isometry desired is $e \mapsto (x \mapsto \langle e,\ \pi(x)\rangle)$. □

Because of this, we may use the fixed space $C({}^{\omega}\{0,\ 1\})$ in place of arbitrary separable Banach spaces E in checking the definition of amenability in the sense of Zimmer. Now there is a projective set in $[0,\ 1]$ which is universal for the class of Borel sets in a Polish space [Mos, 1E.3, p. 43], whence there are projective sets F, C, and Φ parametrizing the Borel fields in $C({}^{\omega}\{0,\ 1\})_1^*$ over $[0,\ 1]$, the cocycles from R to $\mathrm{Iso}(C({}^{\omega}\{0,\ 1\}))$, and the Borel maps from $[0,\ 1]$ to $C({}^{\omega}\{0,\ 1\})_1^*$, respectively. We shall identify the parameter sets with their respective objects. Thus, the statement that $([0,\ 1],\ \mu,\ R)$ is amenable is equivalent to

$$\begin{aligned}\forall A \in F \quad \forall \alpha \in C \quad &\{(\text{ for } \mu\text{-a.e. } t \quad \forall s \in [t] \quad \alpha^*(s,t)A_t = A_s)\\ &\Rightarrow (\exists \varphi \in \Phi \quad [(\text{ for } \mu\text{-a.e. } s \in [0,\ 1] \quad \varphi(s) \in A_s)\\ &\quad \&(\text{ for } \mu\text{-a.e. } t \quad \forall s \in [t] \quad \alpha^*(s,t)\varphi(t) = \varphi(s))]\}.\end{aligned}$$

It remains simply to show that the occurrences of " for μ-a.e. t $\cdots$ " can be replaced by projective formulae. Indeed, in the above case, one can use

$$\forall n\, \exists K \in K([0,\ 1])\, \forall f \in C([0,\ 1])\, \{[(f \leq 1 \quad \&\ f \restriction K = 0) \Rightarrow \textstyle\int f\, d\mu \leq 1/n] \\ \&\forall t \in K\ \cdots \},$$

where $K[0,\ 1]$ is the space of compact subsets of $[0,\ 1]$ in the Hausorff metric. Note that the condition "$[(f \leq 1 \quad \cdots \quad \leq 1/n]$" appearing above is closed in $K[0,\ 1] \times C[0,\ 1]$. Finally, $K[0,\ 1]$ and $C[0,\ 1]$, being Polish, can be replaced by the irrationals in $[0,\ 1]$ for purposes of quantification.

References

[A] Scot Adams, *Trees and amenable equivalence relations*, Ergodic Theory & Dynamical Systems **10** (1990), 1–14.

[AS] Scot Adams and R. Spatzier, *Kazhdan groups, cocycles and trees*, Am. J. Math. **112** (1990), 271–287.

[CFW] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory & Dynamical Systems **1** (1981), 431–450.

[DM] Claude Dellacherie and Paul-André Meyer, *Probabilities and Potential*, Vol. C, North-Holland, Amsterdam, 1988.

[DK] Randall Dougherty and Alexander S. Kechris, *Some remarks on trees and amenable equivalence relations*, unpublished notes, 1988.

[FH] J. Faraut and K. Harzallah, *Distances Hilbertiennes invariantes sur un espace homogène*, Ann. Inst. Fourier, Grenoble **24** (1974), 171–217.

[FM] Jacob Feldman and Calvin C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras, I*, Trans. Am. Math. Soc. **234** (1977), 289–324.

[Gre] F.P. Greenleaf, *Amenable actions of locally compact groups*, J. Functional Analysis **4** (1969), 295–315.

[Gri] Geoffrey Grimmett, *Percolation*, Springer, New York, 1989.

[Gro] M. Gromov, *Rigid transformation groups*, in *Géometrie Differentielle* (D. Bernard and Y. Choquet-Bruhat, eds.), Hermann, Paris, 1988.

[Kec] Alexander S. Kechris, *Amenable equivalence relations and Turing degrees*, J. Symb. Logic **56** (1991), 182–194.

[Kes1] Harry Kesten, *Full Banach mean-values on countable groups*, Math. Scand. **7** (1959), 146–156.

[Kes2] Harry Kesten, *Symmetric random walks on groups*, Trans. Am. Math. Soc. **92** (1959), 336-354.

[L1] Russell Lyons, *Random walks and percolation on trees*, Ann. Prob. **18** (1990), 931–958.

[L2] Russell Lyons *The Ising model and percolation on trees and tree-like graphs*, Commun. Math. Phys. **125** (1989), 337–353.

[L3] Russell Lyons, *Random walks, capacity, and percolation on trees*, Ann. Prob., to appear.

[LP] Russell Lyons and Robin Pemantle, *Random walk in a random environment and first-passage percolation on trees*, Ann. Prob., to appear.

[Mey] Paul-André Meyer, *Limites médiales, d'après Mokobodzki*, Séminaire de Probabilités VII, Université de Strasbourg, Lecture Notes in Mathematics 321, Springer, Berlin, 1973, pp. 198–204.

[Mos] Yiannis N. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam, 1980.

[N] Claudio Nebbia, *Amenability and Kunze-Stein property for groups acting on a tree*, Pacific J. Math. **135** (1988), 371–380.

[P] Alan L.T. Paterson, *Amenability*, American Mathematical Society, Providence, 1988.

[R] Halsey L. Royden, *Real Analysis*, third ed., Macmillan, New York, 1988.

[S] J.-P. Serre, *Trees*, Springer-Verlag, New York, 1980.

[SZ] R. Spatzier and R. Zimmer, *Fundamental groups of negatively curved manifolds and actions of semisimple groups*, Topology, to appear.

[Wat] Y. Watatani, *Property (T) of Kazhdan implies property (FA) of Serre*, Math. Japon. **27** (1981), 97–103.

[Woe] Wolfgang Woess, *Amenable group actions on infinite graphs*, Math. Ann. **284** (1989), 251–265.

[Z1] Robert J. Zimmer, *Hyperfinite factors and amenable ergodic actions*, Invent. Math. **41** (1977), 23–31.

[Z2] Robert J. Zimmer, *Amenable ergodic groups actions and an application to Poisson boundaries of random walks*, J. Functional Analysis **27** (1978), 350–372.

[Z3] Robert J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.

[Z4] Robert J. Zimmer, *Representations of fundamental groups of manifolds with a semisimple transformation group*, J. Am. Math. Soc. **2** (1989), 201–213.

[Z5] Robert J. Zimmer, *Kazhdan groups acting on compact manifolds*, Invent. Math. **75** (1984), 425–436.